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Connection between Yangian symmetry and the quantum inverse scattering method

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Abstract. The quantum nonlinear Schrödinger model with two-component fermions exhibits a Yangian symmetry when considered on an infinite interval. We construct the generators of the Yangian using Dunkl operators. We show an equivalence between the monodromy matrix constructed from the Dunkl operators and that of the quantum inverse scattering method. Under the Yangian algebra the space of states with a fixed particle number forms a tensor product representation of fundamental representations.

1. Introduction

The spin generalizations of the Calogero–Sutherland models [1, 2] and the Haldane–Shastry spin chains [3, 4] have played a prominent role in the recent development of the theory of quantum integrable models. The models were shown [1, 5] to exhibit a Yangian symmetry [6]. This Yangian symmetry can be derived from a representation of the degenerate affine Hecke algebra [1]. One of the important features of these models is that the Yangian symmetry coexists with the periodic boundary condition (PBC), while in other quantum models solved by the Bethe ansatz the Yangian symmetry is broken by the PBC. Even in such cases the Yangian symmetry survives in the limit of an infinite interval. For example, the Hubbard model on the infinite interval has the Yangian symmetry $\mathrm{Y(sl(2))} \oplus \mathrm{Y(sl(2))}$ [7] and similar discussions on the degenerate affine Hecke algebra can be developped [8].

In this paper we consider the quantum nonlinear Schrödinger model with spin-$\frac{1}{2}$ fermions (δ-function fermion gas) with repulsive interaction [9–11]. It possesses a trivial sl(2) symmetry, and when we consider the model on a finite interval the sl(2) symmetry is possibly the only symmetry of it. However, when considered on an infinite interval, the model gains a larger symmetry, the Yangian symmetry Y(sl(2)), and therefore its spectrum is highly degenerate. We construct a representation of the generators of Y(sl(2)) following [1, 8]. First we introduce a representation of the degenerate affine Hecke algebra. The representation [12–14] is expressed in the form of Dunkl operators. Using it we obtain a monodromy matrix $T(u)$ satisfying the exchange relation (sometimes, the Yang–Baxter relation),

$$R_{00'}(u-v)T_0(u)T_{0'}(v) = T_{0'}(v)T_0(u)R_{00'}(u-v) \tag{1.1}$$

where u, v are spectral parameters. Then we find the generators of Y(sl(2)) in the expansion of the monodromy matrix. Since the Hamiltonian of the quantum nonlinear Schrödinger

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model of spin-$\frac{1}{2}$ fermions is found in the quantum determinant of the monodromy matrix, this model is related to the Yangian Y(sl(2)).

On the other hand, there is another monodromy matrix $\tilde{T}(u)$ for the model. It is constructed by the quantum inverse scattering method (QISM) [15, 16]. This monodromy matrix is a 3×3 matrix whose elements are second-quantized operators, while the monodromy matrix made from the Dunkl operators is a 2×2 matrix with elements of first-quantized operators. Though these two matrices are far from alike at first sight, we can show that a 2×2 submatrix of the QISM monodromy matrix is identical with the monodromy matrix made from the Dunkl operators. This is a main result of this paper.

As an application of this remarkable relation, we consider the Yangian representation which n-particle states form. We find that the QISM states of a fixed particle number transform under the Yangian algebra as a tensor product representation of fundamental representations, which is quite natural with the physical meaning of the coproduct. Since this representation is proved to be irreducible, all the states with a fixed particle number can be obtained from the Yangian highest weight state, which is a plane wavestate of up-spin particles only. In this way, we obtain a simple description of particle states.

This paper is organized as follows. In section 2 we present a monodromy matrix satisfying the exchange equations out of a representation of the degenerate affine Hecke algebra. Then the generators of the Yangian Y(sl(2)) are constructed from the monodromy matrix in the usual way. Section 3 relates this Yangian symmetry with the quantum nonlinear Schrödinger model of spin-$\frac{1}{2}$ fermions. In section 4 we explain the QISM for the model on the infinite interval, introducing another monodromy matrix. An equivalence between the two monodromy matrices is established in section 5. As an application we investigate the Yangian representations of particle states in section 6. The last section is devoted to concluding remarks.

2. Representation of the degenerate affine Hecke algebra and the Yangian Y(sl(2))

The degenerate affine Hecke algebra has proved to be quite useful in the theory of quantum solvable models. The degenerate affine Hecke algebra is defined by two sets of generators, $d_i (i = 1, \ldots, n)$ and $K_{ii+1} (i = 1, \ldots, n-1)$, satisfying the following relations;

$$K_{ii+1}K_{i+1i+2}K_{ii+1} = K_{i+1i+2}K_{ii+1}K_{i+1i+2} \tag{2.1}$$

$$K_{ii+1}^2 = 1 \tag{2.2}$$

$$[K_{ii+1}, d_k] = 0 \qquad k \neq i,\ i+1 \tag{2.3}$$

$$K_{ii+1}d_i - d_{i+1}K_{ii+1} = -\mathrm{i}c \tag{2.4}$$

$$[d_i, d_j] = 0 \tag{2.5}$$

where c is a constant. The extra factor $-\mathrm{i}$ on the RHS of equation (2.4) is introduced for later convenience. Clearly the operators $K_{ii+1} (i = 1, \ldots, n-1)$ generate the symmetric group S_n. We consider a representation of this algebra acting on the space of functions of n variables $x_1, \ldots, x_n$. The operator K_{ii+1} is represented by a permutation of variables $\{x_1, \ldots, x_n\}$,

$$(K_{ii+1}f)(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) = f(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n) \tag{2.6}$$

and the operator d_i is represented by the differential operator (Dunkl operator) [12–14];

$$d_i = -\mathrm{i}\frac{\partial}{\partial x_i} - \mathrm{i}c \sum_{j(>i)} \theta(x_j - x_i)K_{ij} + \mathrm{i}c \sum_{j(<i)} \theta(x_i - x_j)K_{ij}. \tag{2.7}$$

Here $\theta(x)$ is the step function;

$$\theta(x) = \begin{cases} 1 & x > 0 \\ \frac{1}{2} & x = 0 \\ 0 & x < 0. \end{cases} \tag{2.8}$$

We can easily verify that these operators satisfy the defining relations (2.1)–(2.5) of the degenerate affine Hecke algebra. From the physical viewpoint we shall regard this representation space as a space of wavefunctions of n-particles with coordinates $x_1, \ldots, x_n$. We assume that each particle has a spin σ_i taking two values $\sigma_i = \uparrow, \downarrow$.

It is convenient to introduce a projection π as

$$\pi(OK_{ij}) = -\pi(O)P_{ij} \tag{2.9}$$

where P_{ij} exchanges the spins of particles i and j and O is an arbitrary operator. In physics, this operation means a projection onto the subspace of fermionic wavefunctions. Note that P_{ij} commutes with K_{kl} and with d_k. We apply the rule (2.9) repeatedly until all K_{ij} are eliminated. Using this projection, we can construct a monodromy matrix which satisfies the exchange relation and preserves the space of fermionic wavefunctions.

The monodromy matrix $T_0(u)$ is defined by

$$\hat{T}_0(u) = \left(1 + \frac{\mathrm{i}cP_{01}}{u - d_1}\right) \ldots \left(1 + \frac{\mathrm{i}cP_{0n}}{u - d_n}\right) \tag{2.10}$$

$$T_0(u) = \pi(\hat{T}_0(u)) \tag{2.11}$$

where the subscript '0' indicates that it operates upon the auxiliary space indexed as zero. The exchange relation (1.1) takes the form;

$$(u - v + \mathrm{i}cP_{00'})T_0(u)T_{0'}(v) = T_{0'}(v)T_0(u)(u - v + \mathrm{i}cP_{00'}). \tag{2.12}$$

A proof of this exchange relation reduces to the following two relations;

$$(u - v + \mathrm{i}cP_{00'})\hat{T}_0(u)\hat{T}_{0'}(v) = \hat{T}_{0'}(v)\hat{T}_0(u)(u - v + \mathrm{i}cP_{00'}) \tag{2.13}$$

$$\pi(\hat{T}_0(u)\hat{T}_{0'}(v)) = \pi(\hat{T}_0(u))\pi(\hat{T}_{0'}(v)) \tag{2.14}$$

equation (2.13) holds since the operators d_i commute with each other and with P_{ij}. For a proof of equation (2.14), it is sufficient to show

$$\pi(K_{ii+1}\hat{T}_{0'}(v)) = -P_{ii+1}\pi(\hat{T}_{0'}(v)) \tag{2.15}$$

which can be proven as in [1]. Then, the monodromy matrix (2.11) with (2.10) satisfies the exchange relation (2.12).

Next we shall construct a representation of the Yangian algebra out of the above monodromy matrix. Since the particles have spin-$\frac{1}{2}$, the Yangian algebra Y(sl(2)) emerges in this case. The Yangian Y(sl(2)) is generated by six generators $Q_k^a (a = 1, 2, 3, k = 0, 1)$ satisfying [6, 17]

$$[Q_0^a, Q_0^b] = f^{abc}Q_0^c \tag{2.16}$$

$$[Q_0^a, Q_1^b] = f^{abc}Q_1^c \tag{2.17}$$

$$\begin{aligned} &[[Q_1^a, Q_1^b], [Q_0^c, Q_1^d]] + [[Q_1^c, Q_1^d], [Q_0^a, Q_1^b]] \\ &\quad = \lambda^2(A^{abkefg}f^{cdk} + A^{cdkefg}f^{abk})\{Q_0^e, Q_0^f, Q_1^g\} \end{aligned} \tag{2.18}$$

where λ is a non-zero constant, $f^{abc} = \mathrm{i}\varepsilon^{abc}$ is the structure constant of sl(2) with ε^{abc} being the totally antisymmetric tensor, and

$$A^{abcdef} = f^{adk}f^{bel}f^{cfm}f^{klm}. \tag{2.19}$$

We use the convention that a summation is performed over any indices appearing twice. The bracket { } in (2.18) indicates the symmetrized product;

$$\{x_1, \ldots, x_m\} = \frac{1}{m!} \sum_{\sigma \in S_m} x_{\sigma_1} \ldots x_{\sigma_m}. \tag{2.20}$$

Using the exchange relation (2.12) we get a representation of the Yangian Y(sl(2)). The generators of Y(sl(2)) ($Q_k^a, k = 0, 1, a = 1, 2, 3$) are found in the expansion of the monodromy matrix with respect to the spectal parameter u;

$$T_0(u) = 1 + \mathrm{i}c \sum_{k=0}^{\infty} \frac{1}{u^{k+1}} (Q_k^0 \mathbf{1} + 2 Q_k^a t_0^a) \tag{2.21}$$

where $2t_0^a$ are the Pauli matrices and $\mathbf{1}$ is the 2×2 identity matrix, both acting on the auxiliary space. Hereafter we shall omit the subscript '0' in $T_0(u)$ unless it is necessary. The constant λ in equations (2.16)–(2.18) is determined as $\lambda = \mathrm{i}c$. The explicit forms of the generators are

$$Q_0^a = \sum_{j=1}^{n} t_j^a \tag{2.22}$$

$$Q_1^a = -\mathrm{i} \sum_{j=1}^{n} t_j^a \frac{\partial}{\partial x_j} + \frac{\mathrm{i}c}{2} \sum_{i \neq j} \varepsilon(x_i - x_j) f^{abc} t_i^b t_j^c + \frac{\mathrm{i}c(n-1)}{2} Q_0^a \tag{2.23}$$

where $2t_j^a$ is the Pauli matrix acting on the jth spin and $\varepsilon(x)$ is the signature function,

$$\varepsilon(x) = \begin{cases} \dfrac{x}{|x|} & x \neq 0 \\ 0 & x = 0. \end{cases} \tag{2.24}$$

3. Conserved operators of the quantum nonlinear Schrödinger model

Given a monodromy matrix satisfying the exchange relation (2.12), we can construct an operator commutable with every component of the monodromy matrix. Such an operator, known as the quantum determinant [15], is given by

$$\mathrm{Det}_q\, T(u) = T_{11}(u - \mathrm{i}c) T_{22}(u) - T_{21}(u - \mathrm{i}c) T_{12}(u) \tag{3.1}$$

where $T_{\alpha\beta}(u) (\alpha, \beta = 1, 2)$ are the elements of the monodromy matrix $T_0(u)$,

$$T(u) = \begin{pmatrix} T_{11}(u) & T_{12}(u) \\ T_{21}(u) & T_{22}(u) \end{pmatrix}. \tag{3.2}$$

It is easily evaluated to be of the following compact form;

$$\mathrm{Det}_q\, T(u) = \pi\left(\frac{\hat{\Delta}_n(u + \mathrm{i}c)}{\hat{\Delta}_n(u)}\right) = \frac{\Delta_n(u + \mathrm{i}c)}{\Delta_n(u)} \tag{3.3}$$

where $\hat{\Delta}_n(u) = \prod_{j=1}^{n} (u - d_j)$ and $\Delta_n(u) = \pi(\hat{\Delta}_n(u))$. A straightforward calculation also shows that the operator $\hat{\Delta}_n(u)$ commutes with K_{ii+1}, P_{ii+1} and d_i, and then we get

$$[\hat{\Delta}_n(u), \hat{\Delta}_n(v)] = 0 \tag{3.4}$$

$$[\Delta_n(u), T_{\alpha\beta}(v)] = 0 \tag{3.5}$$

$$[\Delta_n(u), \Delta_n(v)] = 0. \tag{3.6}$$

We expand $\Delta_n(u)$ as

$$\Delta_n(u) = \sum_{j=0}^{n} (-1)^j C_j u^{n-j} = C_0 u^n - C_1 u^{n-1} + \cdots + (-1)^n C_n. \tag{3.7}$$

Note that C_j and $C_i (i \neq j)$ commute due to (3.6). The first three of C_j are calculated as

$$C_0 = 1 \qquad C_1 = \hat{P} \qquad C_2 = \tfrac{1}{2}(\hat{P}^2 - \hat{H}) \tag{3.8}$$

where

$$\hat{P} = -\mathrm{i} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \tag{3.9}$$

$$\hat{H} = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \sum_{1 \leqslant i < j \leqslant n} 2c\delta(x_i - x_j). \tag{3.10}$$

Assuming c to be real, we notice that the operator $\hat{H}$ is the first-quantized version of the Hamiltonian of the quantum nonlinear Schrödinger model for two-component fermions. The operator $\hat{P}$ indicates the total momentum. Therefore, we have obtained n conserved operators $C_1, \ldots, C_n$ of the quantum nonlinear Schrödinger model for two-component fermions including its Hamiltonian. Since they are in involution, $[C_i, C_j] = 0$, the model is integrable. From equation (3.5) we deduce

$$[\hat{H}, Q_0^a] = 0 \qquad [\hat{H}, Q_1^a] = 0 \tag{3.11}$$

$$[\hat{P}, Q_0^a] = 0 \qquad [\hat{P}, Q_1^a] = 0. \tag{3.12}$$

Relations in (3.11) show that the model exhibits the Y(sl(2)) symmetry.

To conclude this section, we shall perform the second-quantization of the above operators in order to relate them to the QISM operators. Let $\phi_\alpha(x)(\alpha = \uparrow, \downarrow)$ denote fermionic field operators satisfying canonical anticommutation relations;

$$\begin{aligned} &[\phi_\alpha(x), \phi_\beta^\dagger(y)]_+ = \delta_{\alpha\beta}\delta(x - y) \\ &[\phi_\alpha(x), \phi_\beta(y)]_+ = 0 \qquad (\alpha, \beta = \uparrow, \downarrow). \end{aligned} \tag{3.13}$$

The vacuum $|0\rangle$ is defined as $\phi_\alpha(x)|0\rangle = 0$. The first three conserved operators for the quantum nonlinear Schrödinger model of spin-$\frac{1}{2}$ fermions are given by the following expression;

$$\hat{N} = \int \mathrm{d}x\, \phi_\alpha^\dagger(x)\phi_\alpha(x) \qquad \text{particle number} \tag{3.14}$$

$$\hat{P} = -\mathrm{i} \int \mathrm{d}x\, \phi_\alpha^\dagger(x) \frac{\partial}{\partial x} \phi_\alpha(x) \qquad \text{total momentum} \tag{3.15}$$

$$\hat{H} = \int \mathrm{d}x\, \left\{ \frac{\partial \phi_\alpha^\dagger}{\partial x} \frac{\partial \phi_\alpha}{\partial x} + c\phi_\beta^\dagger \phi_\alpha^\dagger \phi_\alpha \phi_\beta \right\} \qquad \text{Hamiltonian.} \tag{3.16}$$

We can also derive the second-quantized form of Yangian generators as

$$Q_0^a = \int \mathrm{d}x t_{\beta\alpha}^a \phi_\beta^\dagger(x)\phi_\alpha(x) \tag{3.17}$$

$$\begin{aligned} Q_1^a = &-\mathrm{i} \int \mathrm{d}x\, t_{\beta\alpha}^a \phi_\beta^\dagger \frac{\partial \phi_\alpha(x)}{\partial x} \\ &- \frac{\mathrm{i}c}{2} \int \mathrm{d}x\, \mathrm{d}y\, \varepsilon(y - x) t_{\beta\sigma}^a \phi_\beta^\dagger(x) \phi_\alpha^\dagger(y) \phi_\alpha(x) \phi_\sigma(y) + \frac{\mathrm{i}c}{2}(\hat{N} - 1) Q_0^a \end{aligned} \tag{3.18}$$

where $2t^a$ are the Pauli matrices.

4. QISM of the nonlinear Schrödinger model on the infinite interval

We shall investigate the connection between the Yangian symmetry and the QISM. The QISM for the model (3.16) is formulated as follows [16]. Hereafter we assume $c > 0$, i.e. the repulsive interaction. The auxiliary linear equation reads as

$$\frac{\partial}{\partial x}\tilde{T}(x,y|k) =: L(x,k)\tilde{T}(x,y|k) : \tag{4.1}$$

$$\tilde{T}(x,y|k)|_{x=y} = I_3 \tag{4.2}$$

$$L(x,k) = \begin{bmatrix} \mathrm{i}\frac{k}{2} & 0 & \mathrm{i}\sqrt{c}\phi_\uparrow(x) \\ 0 & \mathrm{i}\frac{k}{2} & \mathrm{i}\sqrt{c}\phi_\downarrow(x) \\ -\mathrm{i}\sqrt{c}\phi_\uparrow^\dagger(x) & -\mathrm{i}\sqrt{c}\phi_\downarrow^\dagger(x) & -\mathrm{i}\frac{k}{2} \end{bmatrix}. \tag{4.3}$$

Here : : indicates the normal ordering and I_3 is the 3×3 unit matrix. We regard 3×3 matrices appearing here as supermatrices where the elements $(1,1), (1,2), (2,1), (2,2), (3,3)$ are bosonic operators and the others are fermionic ones. Then the monodromy matrix $\tilde{T}(k)$ on the infinite interval is defined as

$$\tilde{T}(k) = \lim_{\substack{x\to\infty \\ y\to-\infty}} \begin{bmatrix} \mathrm{e}^{-\mathrm{i}\frac{k}{2}x} & 0 & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\frac{k}{2}x} & 0 \\ 0 & 0 & \mathrm{e}^{\mathrm{i}\frac{k}{2}x} \end{bmatrix} \tilde{T}(x,y|k) \begin{bmatrix} \mathrm{e}^{\mathrm{i}\frac{k}{2}y} & 0 & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\frac{k}{2}y} & 0 \\ 0 & 0 & \mathrm{e}^{-\mathrm{i}\frac{k}{2}y} \end{bmatrix}. \tag{4.4}$$

We express the elements of $\tilde{T}(k)$ as

$$\tilde{T}(k) = \begin{bmatrix} A_{\uparrow\uparrow}(k) & A_{\uparrow\downarrow}(k) & B_\uparrow(k) \\ A_{\downarrow\uparrow}(k) & A_{\downarrow\downarrow}(k) & B_\downarrow(k) \\ C_\uparrow(k) & C_\downarrow(k) & D(k) \end{bmatrix}. \tag{4.5}$$

It is possible to derive the commutation rules among the operators $A_{\alpha\beta}(k)$, $B_\alpha(k)$, $C_\alpha(k)$, $D(k)$. Let $\otimes_s$ denote the tensor product in the sense of supermatrices, i.e.

$$(A\otimes_s B)_{ij,kl} = A_{ik}B_{jl}(-1)^{(p(i)+p(k))p(j)} \tag{4.6}$$

where $p(1) = p(2) = 1$, $p(3) = 0$. The monodromy matrix $\tilde{T}(u)$ satisfies the following generalized exchange relation [16];

$$R_+(u-v)[\tilde{T}(u)\otimes_s \tilde{T}(v)] = [\tilde{T}(v)\otimes_s \tilde{T}(u)]R_-(u-v) \tag{4.7}$$

where

$$\begin{aligned} R_\pm(u-v) = {} & \sum_{i,j=1}^{2}\left(\frac{1}{u-v}\frac{-\mathrm{i}c}{u-v-\mathrm{i}c}e^{ii}\otimes e^{jj} + \frac{-1}{u-v-\mathrm{i}c}e^{ij}\otimes e^{ji}\right) \\ & \pm\mathrm{i}\pi\delta(u-v)\sum_{i=1}^{2}(e^{ii}\otimes e^{33} - e^{33}\otimes e^{ii}) + \frac{u-v+\mathrm{i}c}{(u-v+\mathrm{i}\varepsilon)^2}\sum_{i=1}^{2}e^{i3}\otimes e^{3i} \\ & + \frac{1}{u-v-\mathrm{i}c}\sum_{i=1}^{2}e^{3i}\otimes e^{i3} + \frac{1}{u-v}e^{33}\otimes e^{33}. \end{aligned} \tag{4.8}$$

In the above, e^{ij} is a 3×3 matrix with elements $(e^{ij})_{pq} = \delta_{ip}\delta_{jq}$ and $\otimes$ is a matrix tensor product in the usual sense. Taking matrix elements from equation (4.7), some of the commutation relations, which will be used in later discussions, are

$$D(k)D(p) = D(p)D(k) \tag{4.9}$$

$$C_\alpha(k)D(p) = \frac{k-p}{k-p-\mathrm{i}c}D(p)C_\alpha(k) \tag{4.10}$$

$$C_\alpha(k)C_\beta(p) = -\frac{k-p}{k-p-\mathrm{i}c}C_\beta(p)C_\alpha(k) - \frac{\mathrm{i}c}{k-p-\mathrm{i}c}C_\alpha(p)C_\beta(k). \tag{4.11}$$

Relation (4.9) implies that $D(k)$ is a generator of conserved operators including the Hamiltonian $\hat{H}$. Indeed, the expansion of $D(k)$ in powers of k^{-1} yields

$$D(k) = 1 + \frac{\mathrm{i}c\hat{N}}{k} + \frac{1}{k^2}\{\mathrm{i}c\hat{P} - \frac{c^2}{2}\hat{N}(\hat{N}-1)\}$$
$$+\frac{1}{k^3}\left\{\mathrm{i}c\hat{H} - c^2\hat{P}(\hat{N}-1) - \frac{\mathrm{i}c^3}{6}\hat{N}(\hat{N}-1)(\hat{N}-2)\right\} + \mathrm{O}\left(\frac{1}{k^4}\right). \tag{4.12}$$

Explicit forms of the operators $\hat{N}$, $\hat{P}$ and $\hat{H}$ have been given in (3.14)–(3.16).

We now define an operator $R^\dagger_\alpha(k)$, which is a quantum analogue of the reflection coefficient, as

$$R^\dagger_\alpha(k) = \frac{\mathrm{i}}{\sqrt{c}} C_\alpha(k) D^{-1}(k) \qquad \alpha = \uparrow, \downarrow\ . \tag{4.13}$$

The operator $R^\dagger_\alpha(k)$ satisfies the commutation relations,

$$[\hat{N}, R^\dagger_\alpha(k)] = R^\dagger_\alpha(k) \tag{4.14}$$
$$[\hat{P}, R^\dagger_\alpha(k)] = kR^\dagger_\alpha(k) \tag{4.15}$$
$$[\hat{H}, R^\dagger_\alpha(k)] = k^2R^\dagger_\alpha(k). \tag{4.16}$$

These relations indicate that this operator plays a role of the creation operator of a quasiparticle with momentum k and energy k^2.

5. Second-quantization of the monodromy matrix

From the second-quantized forms of $\mathrm{Det}_q\, T(u)$, Q^a_0 and Q^a_1 in section 3, we can in principle express Q^0_k, Q^a_k and the monodromy matrix $T(u)$ in a second-quantized form (see the appendix). Then, the following theorem holds.

Theorem 1.

$$T_{\alpha\beta}(u) = A_{\alpha\beta}(u) \tag{5.1}$$
$$\mathrm{Det}_q\, T(u) = \mathrm{Det}_q\, A(u) = D(u) \tag{5.2}$$

where $T(u)$ is regarded as a second-quantized form of the monodromy matrix (2.11).

These relations are quite surprising since $T(u)$ and $A(u)$ are far from alike. For a proof of the theorem there is no need to second-quantize the full formula of $T(u)$. The proof goes as follows. First we see that $T_{\alpha\beta}(u)$ and $A_{\alpha\beta}(u)$ satisfy the same exchange relation (2.12). This indicates that two Y(sl(2)) representations are derived respectively from these two monodromy matrices. Next we check that Q^a_0, Q^a_1 and the quantum determinant obtained from $T_{\alpha\beta}(u)$ and those from $A_{\alpha\beta}(u)$ are equal. Since the Yangian Y(gl(2)) is generated by Q^a_0, Q^a_1, and the quantum determinant (see the appendix), we conclude that the full formulae for $T_{\alpha\beta}(u)$ and $A_{\alpha\beta}(u)$ are equal.

Proof. To begin with, we see that $A_{\alpha\beta}(u)$ and $T_{\alpha\beta}(u)$ obey the same commutation relations;

$$(u-v)[A_{\beta\gamma}(u), A_{\alpha\delta}(v)] = \mathrm{i}c\{A_{\alpha\gamma}(v)A_{\beta\delta}(u) - A_{\alpha\gamma}(u)A_{\beta\delta}(v)\} \tag{5.3}$$
$$(u-v)[T_{\beta\gamma}(u), T_{\alpha\delta}(v)] = \mathrm{i}c\{T_{\alpha\gamma}(v)T_{\beta\delta}(u) - T_{\alpha\gamma}(u)T_{\beta\delta}(v)\}. \tag{5.4}$$

Equation (5.4) is equivalent to the exchange relation (2.12), and equation (5.3) is obtained by extracting a submatrix of a generalized exchange relation (4.7).

Secondly we compare the two sets of Y(sl(2)) generators derived respectively from $A_{\alpha\beta}(u)$ and $T_{\alpha\beta}(u)$. By partial integrations of $A_{\alpha\beta}(k)$;

$$A_{\alpha\beta}(k) = \delta_{\alpha\beta} + \sum_{n=1}^{\infty}(-c)^n \int \prod_{j=1}^{n} \mathrm{d}y_j \prod_{j=1}^{n} \mathrm{d}z_j\, \theta(z_n < y_n < \cdots < z_1 < y_1)$$
$$\times \phi^{\dagger}_{\gamma_1}(z_1) \ldots \phi^{\dagger}_{\gamma_{n-1}}(z_{n-1}) \phi^{\dagger}_{\beta}(z_n) \phi_{\gamma_{n-1}}(y_n) \ldots \phi_{\gamma_1}(y_2)\phi_{\alpha}(y_1) \mathrm{e}^{\mathrm{i}k(\sum z - \sum y)} \qquad (5.5)$$

we get

$$A_{\alpha\beta}(k) = \delta_{\alpha\beta} + \frac{ic}{k} \int \mathrm{d}x\, \phi^{\dagger}_{\beta}(x)\phi_{\alpha}(x) + \frac{1}{k^2}\left\{ c \int \mathrm{d}x\, \phi^{\dagger}_{\beta}(x) \frac{\partial}{\partial x}\phi_{\alpha}(x) \right.$$
$$\left. -c^2 \int \mathrm{d}x \int \mathrm{d}y\, \theta(x-y) \phi^{\dagger}_{\gamma}(x) \phi^{\dagger}_{\beta}(y) \phi_{\gamma}(y) \phi_{\alpha}(x) \right\} + \mathrm{O}\left(\frac{1}{k^3}\right). \qquad (5.6)$$

Since the matrix $A_{\alpha\beta}(u)$ satisfies an exchange relation (2.12), we can deduce a representation of Y(sl(2)) from $A_{\alpha\beta}(u)$ as well. We put

$$A(u) = 1 + \mathrm{i}c \sum_{k=0}^{\infty} \frac{1}{u^{k+1}} (\tilde{Q}^0_k \mathbf{1} + 2\tilde{Q}^a_k t^a) \qquad (5.7)$$

where $A(u)$ is regarded as a 2×2 matrix with elements $A_{\alpha\beta}(u)$. Then it follows that

$$\tilde{Q}^a_0 = \int \mathrm{d}x\, t^a_{\beta\alpha} \phi^{\dagger}_{\beta}(x) \phi_{\alpha}(x) = Q^a_0 \qquad (5.8)$$

$$\tilde{Q}^a_1 = -\mathrm{i} \int \mathrm{d}x\, t^a_{\beta\alpha} \phi^{\dagger}_{\beta}(x) \frac{\partial}{\partial x} \phi_{\alpha}(x) - \frac{\mathrm{i}c}{2} \int \mathrm{d}x\, \mathrm{d}y\, \varepsilon(y-x) t^a_{\beta\sigma} \phi^{\dagger}_{\beta}(x) \phi^{\dagger}_{\alpha}(y) \phi_{\alpha}(x) \phi_{\sigma}(y)$$
$$+ \frac{\mathrm{i}c}{2}(\hat{N}-1)\tilde{Q}^a_0 = Q^a_1. \qquad (5.9)$$

Therefore, two representations of Y(sl(2)) deduced respectively from $A_{\alpha\beta}(u)$ and from $T_{\alpha\beta}(u)$ are identical.

Finally we check that the two quantum determinants calculated from $A_{\alpha\beta}(u)$ and $T_{\alpha\beta}(u)$ are identical, i.e. equation (5.2) holds. into We can show that the three operators in equation (5.2) acting on the n-particle state

$$|\Psi\rangle = R^{\dagger}_{i_1}(k_1) \ldots R^{\dagger}_{i_n}(k_n)|0\rangle \qquad (5.10)$$

give the same eigenvalue;

$$D(u)|\Psi\rangle = \prod_{j=1}^{n} \frac{u + \mathrm{i}c - k_j}{u - k_j} |\Psi\rangle \qquad (5.11)$$

$$\mathrm{Det}_q\, A(u)|\Psi\rangle = \prod_{j=1}^{n} \frac{u + \mathrm{i}c - k_j}{u - k_j} |\Psi\rangle \qquad (5.12)$$

$$\mathrm{Det}_q\, T(u)|\Psi\rangle = \prod_{j=1}^{n} \frac{u + \mathrm{i}c - k_j}{u - k_j} |\Psi\rangle \qquad (5.13)$$

which shows the validity of equation (5.2). Note that the n-particle states (5.10) form a complete set [11]. It is easy to show equations (5.11) and (5.12) using the commutation relation between the QISM operators. And we have explicitly shown equation (5.13) in the case $k_1, \ldots, k_n$ are all distinct. We shall only sketch the outline of the proof of (5.13) since the proof itself is quite complicated and inessential for later discussions. We first explicitly calculate the coordinate wavefunction $\Psi(x_1, \ldots, x_n)$ out of $|\Psi\rangle$. It has a well known Bethe ansatz form. Then we decompose it into eigenstates of $d_j (k = 1, \ldots, n)$ and obtain the following;

$$\Psi(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} \Psi^\sigma(x_1, \ldots, x_n) \tag{5.14}$$

$$d_j \Psi^\sigma(x_1, \ldots, x_n) = k_{\sigma j} \Psi^\sigma(x_1, \ldots, x_n) \qquad (j = 1, \ldots, n). \tag{5.15}$$

Then equation (5.11) follows from the fact that $\mathrm{Det}_q T(u)$ is a symmetric function of $d_1, \ldots, d_n$ (see equation (3.3)). Thus, we have shown theorem 1. □

Utilizing the above result, we calculate commutators between the Yangian generators and the QISM operators $C_\alpha(k), D(k)$. For that purpose we make use of the following commutation rules for the QISM operators;

$$[A_{\alpha\beta}(k), C_\gamma(p)] = \frac{\mathrm{i}c}{k-p} C_\beta(p) A_{\alpha\gamma}(k) \tag{5.16}$$

$$[A_{\alpha\beta}(k), D(p)] = 0. \tag{5.17}$$

The commutators are calculated as

$$[Q_1^a, D(k)] = 0 = [Q_0^a, D(k)] \tag{5.18}$$

$$[Q_1^a, C_\alpha(k)] = k t^a_{\beta\alpha} C_\beta(k) - \mathrm{i}c f^{abc} t^b_{\beta\alpha} C_\beta(k) Q_0^c + \frac{\mathrm{i}c}{2} C_\beta(k) \hat{N} t^a_{\beta\alpha} + \frac{\mathrm{i}c}{2} C_\alpha(k) Q_0^a \tag{5.19}$$

$$[Q_0^a, C_\alpha(k)] = t^a_{\beta\alpha} C_\beta(k). \tag{5.20}$$

Recall that $D(k)$ is a generator of conserved operators including the particle number $\hat{N}$, the total momentum $\hat{P}$, and the Hamiltonian $\hat{H}$. Of course equation (5.18) implies equations (3.11) and (3.12) again. By putting

$$Q_1^{a'} = Q_1^a - \frac{\mathrm{i}c}{2}(\hat{N} - 1) Q_0^a \tag{5.21}$$

the commutation rule (5.19) is cast into a simpler form

$$[Q_1^{a'}, C_\alpha(k)] = k t^a_{\beta\alpha} C_\beta(k) - \mathrm{i}c f^{abc} t^b_{\beta\alpha} C_\beta(k) Q_0^c. \tag{5.22}$$

This redefinition ($Q_1^a \to Q_1^{a'}$) does not affect the defining relations of the Yangian. If the constant c is real, all the generators of the Yangian Y(sl(2)) become Hermitian by the redefinition. Using equations (5.18)–(5.20) we deduce

$$\begin{aligned} &[Q_1^{a'}, R^\dagger_\alpha(k)] = k t^a_{\beta\alpha} R^\dagger_\beta(k) - \mathrm{i}c f^{abc} t^b_{\beta\alpha} R^\dagger_\beta(k) Q_0^c \\ &[Q_0^a, R^\dagger_\alpha(k)] = t^a_{\beta\alpha} R^\dagger_\beta(k). \end{aligned} \tag{5.23}$$

These relations play an essential role in discussing the representations of the Yangian in the next section.

6. Yangian representations

We note that the vacuum $|0\rangle$ is invariant under this Yangian representation;

$$Q_0^a |0\rangle = 0 \qquad Q_1^{a'} |0\rangle = 0. \tag{6.1}$$

Let n denote the number of particles. We can consider that the space of n-particle states is spanned by the states of n quasiparticles $R^\dagger_{\alpha_1}(k_1) \ldots R^\dagger_{\alpha_n}(k_n)|0\rangle (\alpha_j = \uparrow, \downarrow)$. The quasimomenta $\{k_i\}$ should be real since imaginary quasimomenta cause a divergence of a wavefunction at $|x| \to \infty$.

In the $n = 1$ sector we have

$$Q_0^a R_\alpha^\dagger(k)|0\rangle = t^a_{\beta\alpha} R_\beta^\dagger(k)|0\rangle \tag{6.2}$$

$$Q_1^{a'} R_\alpha^\dagger(k)|0\rangle = k t^a_{\beta\alpha} R_\beta^\dagger(k)|0\rangle. \tag{6.3}$$

The action of $Q_1^{a'}$ on the 1-particle states is k times that of Q_0^a. We call this representation the fundamental representation $W_1(k)$, following Chari and Pressley [18].

In the $n = 2$ sector we get

$$Q_0^a R_\alpha^\dagger(k_1) R_\sigma^\dagger(k_2)|0\rangle = (t^a_{\beta\alpha}\delta_{\rho\sigma} + \delta_{\beta\alpha} t^a_{\rho\sigma}) R_\beta^\dagger(k_1) R_\rho^\dagger(k_2)|0\rangle \tag{6.4}$$

$$Q_1^{a'} R_\alpha^\dagger(k_1) R_\sigma^\dagger(k_2)|0\rangle = (k_1 t^a_{\beta\alpha}\delta_{\rho\sigma} + k_2 \delta_{\beta\alpha} t^a_{\rho\sigma} - \mathrm{i}c f^{abc} t^b_{\beta\alpha} t^c_{\rho\sigma}) R_\beta^\dagger(k_1) R_\rho^\dagger(k_2)|0\rangle. \tag{6.5}$$

This representation is easily identified as a tensor product representation $W_1(k_1) \otimes W_1(k_2)$, where the comultiplication Δ is defined as

$$\Delta(Q_0^a) = Q_0^a \otimes 1 + 1 \otimes Q_0^a \tag{6.6}$$

$$\Delta(Q_1^{a'}) = Q_1^{a'} \otimes 1 + 1 \otimes Q_1^{a'} - \mathrm{i}c f^{abc} Q_0^b \otimes Q_0^c. \tag{6.7}$$

Similarly, the n-particle states $R_{\alpha_1}^\dagger(k_1)\ldots R_{\alpha_n}^\dagger(k_n)|0\rangle(\alpha_j = \uparrow, \downarrow)$ transform under the Yangian Y(sl(2)) as a tensor product representation

$$W_1(k_1) \otimes \cdots \otimes W_1(k_n). \tag{6.8}$$

Since the quasimomenta $\{k_i\}$ are all real, these representations are irreducible under Y(sl(2)), [18] while they are not under the subalgebra sl(2).

Because of this irreducibility, using Yangian generators we can construct all the n-particle states out of the Yangian highest weight state

$$R_\uparrow^\dagger(k_1)\ldots R_\uparrow^\dagger(k_n)|0\rangle. \tag{6.9}$$

By induction we can show that (6.9) is equal to

$$\phi_\uparrow^\dagger(k_1)\ldots\phi_\uparrow^\dagger(k_n)|0\rangle. \tag{6.10}$$

This equality is quite natural since the repulsive contact interaction never occurs between up-spin particles due to the Pauli principle. In conclusion, all the n-particle states can be constructed out of the state (6.10) by using the Yangian generators (3.17) and (3.18). The interesting point is that these three formulae are written in terms of $\phi_i(k)$, not $R_i(k)$. Therefore, in order to describe all the n-particle states, there is no need to use $R_i(k)$.

7. Concluding remarks

In the present paper we have dealt with the quantum nonlinear Schrödinger model with two-component fermions. We think it is possible to extend the above discussions to the multicomponent quantum nonlinear Schrödinger models and to other models solvable by the QISM. A wide applicability of this method would give us a deeper insight into the quantum solvable models. We make a remark on a particularity of the Hubbard model. A Yangian symmetry was found [7] for the Hubbard model. A similar analysis using the degenerate affine Hecke algebra is possible, but its Hamiltonian appears in $\Delta_n(u)$ in the form separated into the left- and the right-hopping parts [8]. It prevents us from the further investigation of the Hubbard model along this method.

We have discussed the connection between the Yangian symmetry and the QISM. Especially, equations (5.1) and (5.2) are quite interesting since the two monodromy matrices $T(u)$ and $\tilde{T}(u)$ ($A(u)$ is a submatrix of $\tilde{T}(u)$) have completely different origins. This interesting correspondence should be a consequence of a more profound structure of the model, which is left as a future problem. On the analogy with this model, we suggest that in other integrable models the QISM on the infinite interval would present an infinite-dimensional symmetry, e.g. the Yangian symmetry or the quantum affine algebra symmetry.

We have also presented a Y(sl(2)) representation of the particle states of the model. As noted in the previous section, the irreducibility of the Yangian representations of the particle states allows us a simple description of the particle states, which we believe is useful for further investigations of the model.

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Appendix. Derivation of Y(sl(2)) from the exchange relation

We begin with the exchange relation for the 2×2 transfer matrix $T(u)$,

$$((u-v) + \hbar P_{00'})T_0(u)T_{0'}(v) = T_{0'}(v)T_0(u)((u-v) + \hbar P_{00'}) \tag{A.1}$$

where P denotes the permutation operator defined by $P(x \otimes y) = y \otimes x$. Extracting the component of the above relation, we get

$$(u-v)[T^{\alpha\beta}(u), T^{\gamma\rho}(v)] = \hbar[T^{\gamma\beta}(v)T^{\alpha\rho}(u) - T^{\gamma\beta}(u)T^{\alpha\rho}(v)]. \tag{A.2}$$

Assuming that $\lim_{|u|\to\infty} T(u) = \mathbf{1}$, we expand $T(u)$ by u^{-1};

$$\begin{aligned} T(u) &= \mathbf{1} + \frac{\hbar}{u}\sum_{n=0}^{\infty}\frac{1}{u^n}T_{n+1} \\ &= \mathbf{1} + \frac{\hbar}{u}\sum_{n=0}^{\infty}\frac{1}{u^n}(Q_n^0\mathbf{1} + Q_n^a\sigma^a) \end{aligned} \tag{A.3}$$

where $\sigma^a (a = 1, 2, 3)$ denote the Pauli matrices and we adopt the convention that we take a summation over the repeated indices. Substituting it to equation (A.2), we have

$$[T_1^{\alpha\beta}, T_n^{\gamma\rho}] = \delta^{\alpha\rho}T_n^{\gamma\beta} - \delta^{\gamma\beta}T_n^{\alpha\rho} \tag{A.4}$$

$$[T_{m+1}^{\alpha\beta}, T_n^{\gamma\rho}] - [T_m^{\alpha\beta}, T_{n+1}^{\gamma\rho}] = \hbar(T_n^{\gamma\beta}T_m^{\alpha\rho} - T_m^{\gamma\beta}T_n^{\alpha\rho}). \tag{A.5}$$

In order to find the Y(sl(2)) structure we change the matrix basis into $\{\mathbf{1}, \sigma^a\}$ and we get

$$[Q_0^a, Q_n^0] = 0 \tag{A.6}$$

$$[Q_0^a, Q_n^b] = f^{abc}Q_n^c \tag{A.7}$$

$$[Q_m^0, Q_n^0] = 0 \tag{A.8}$$

$$[Q_m^a, Q_n^a] = 0 \tag{A.9}$$

$$[Q_m^0, Q_n^a] = [Q_n^0, Q_m^a] \tag{A.10}$$

$$[Q_m^b, Q_n^c] = [Q_n^b, Q_m^c] \tag{A.11}$$

$$[Q_{m+1}^b, Q_n^c] - [Q_m^b, Q_{n+1}^c] = \hbar f^{bcd}(Q_n^d Q_m^0 - Q_m^d Q_n^0) \tag{A.12}$$

$$[Q_{m+1}^0, Q_n^a] - [Q_m^0, Q_{n+1}^a] = \frac{\hbar}{2}f^{abc}(Q_n^b Q_m^c - Q_m^b Q_n^c). \tag{A.13}$$

From these equations we can derive a recursion formula for Q_n^a;

$$Q_n^a = -f^{abc} Q_1^b Q_{n-1}^c + \hbar (Q_0^a Q_{n-1}^0 - Q_{n-1}^a Q_0^0). \tag{A.14}$$

A recursion formula for Q_n^0 should be studied separately since it is related to the existence of a centre of the whole algebra.

It is well known that the quantum determinant,

$$\mathrm{Det}_q\, T(u) = T^{11}(u) T^{22}(u-\hbar) - T^{12}(u) T^{21}(u-\hbar) \tag{A.15}$$

commutes with every component of the transfer matrix $T(u)$. Thus, the coefficients of the expansion

$$\mathrm{Det}_q\, T(u) = 1 + \frac{\hbar}{u} \sum_{n=0}^{\infty} \frac{1}{u^n} a_n \tag{A.16}$$

belongs to the centre of the algebra. The first few of them are

$$a_0 = 2Q_0^0 \tag{A.17}$$

$$a_1 = 2Q_1^0 - \hbar Q_0^a Q_0^a + \hbar Q_0^0 Q_0^0 + \hbar Q_0^0 \tag{A.18}$$

$$a_2 = 2Q_2^0 + \hbar^2 Q_0^0 + 2\hbar Q_1^0 + \hbar^2 (Q_0^0 Q_0^0 - Q_0^a Q_0^a) + \hbar (Q_0^0 Q_1^0 - Q_0^a Q_1^a). \tag{A.19}$$

The general formula for a_n is

$$a_n = 2Q_n^0 + \sum_{m=0}^{n-1} \begin{pmatrix} n \\ n-m \end{pmatrix} \hbar^{n-m} Q_m^0 + \sum_{q=0}^{n-1} \sum_{p=0}^{n-q-1} \begin{pmatrix} n-q-1 \\ p \end{pmatrix} \hbar^{n-p-q} (Q_q^0 Q_p^0 - Q_q^a Q_p^a). \tag{A.20}$$

We can rewrite this equation in the form of the recursion formula for Q_n^0;

$$\begin{aligned} Q_n^0 = \frac{1}{2} a_n - \frac{1}{2} \sum_{m=0}^{n-1} \begin{pmatrix} n \\ n-m \end{pmatrix} \hbar^{n-m} Q_m^0 \\ - \frac{1}{2} \sum_{q=0}^{n-1} \sum_{p=0}^{n-q-1} \begin{pmatrix} n-q-1 \\ p \end{pmatrix} \hbar^{n-p-q} (Q_q^0 Q_p^0 - Q_q^a Q_p^a). \end{aligned} \tag{A.21}$$

Using equations (A.14) and (A.21) we can express Q_n^a and Q_n^0 in terms of Q_0^a, Q_1^a and a_n. The algebra generated by Q_0^a, Q_1^a and a_n is called the Y(gl(2)) Yangian, and that generated by Q_0^a and Q_1^a is called the Y(sl(2)) Yangian. From equations (A.14) and (A.21), a knowledge of Q_0^a, Q_1^a, a_n is sufficient for calculating the whole formula of $T(u)$.

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